JOURNAL OF APPROXIMATION THEORY 23, 1–28 (1978)

On pLg-Splines

P. COPLEY

Department of Mathematics, Baylor University, Waco, Texas 76703

AND

L. L. SCHUMAKER

Center for Numerical Analysis, University of Texas, Austin Texas 78712

Communicated by O. Shisha

Received October 25, 1976

pLg-splines are obtained as the solution of best interpolation problems where the smoothness is measured in terms of the L_p-norm of Lf, L being a linear differential operator. The emphasis of this paper is on the structure and characterization of such splines. Special attention is paid to interpolation defined by linear functionals with support at more than one point. Optimal extension problems, histosplines, and continuous inequality constraints are all treated.

1. INTRODUCTION

We begin by defining the class of Lg-splines of interest in this paper. Let $-\infty < a < b < \infty$, $1 < p < \infty$, and let m be a positive integer. Let

$$L_p^m[a, b] = \{f \in C^{m-1}[a, b] : f^{(m-1)} \text{ be absolutely continuous and } f^{(m)} \in L_p[a, b]\}. \tag{1.1}$$

Let L be a linear differential operator of the form

$$Lf = \sum_0^m a_j f^{(j)}, \qquad a_m(x) \neq 0 \text{ on } [a, b], \qquad a_j \in C^j[a, b], \qquad j = 0,..., m. \tag{1.2}$$

Suppose A is a (possibly infinite) index set, and that $\Lambda = \{\lambda_\alpha\}_{\alpha \in A}$ is a linearly independent collection of bounded linear functionals on $L_p^m[a, b]$. We suppose Λ is uniformly bounded; i.e., there exists $1 \leqslant C < \infty$ such that

$$\| \lambda_\alpha \| = \sup_{f \in L_p^m} \frac{| \lambda_\alpha f |}{\| f \|_{L_p^m}} < C < \infty, \qquad \text{all } \alpha \in A, \tag{1.3}$$

0021-9045/78/0231–0001$02.00/0

where

$$\|f\|_{L_p^m[a,b]} = \|f^{(m)}\|_{L_p[a,b]} + \sum_0^{m-1} |f^{(j)}(a)|.$$

Given real numbers $\underline{y}_\alpha \leqslant \bar{y}_\alpha$ for all $\alpha \in A$, we define

$$U(\Lambda, \underline{y}, \bar{y}) = \{f \in L_p{}^m[a, b]: \underline{y}_\alpha \leqslant \lambda_\alpha f \leqslant \bar{y}_\alpha\,,\ \alpha \in A\}. \tag{1.4}$$

DEFINITION 1.1. Let L and U be as above. Then a function $s \in U$ which satisfies

$$\| Ls \|_p = \inf_{u \in U} \| Lu \|_p \tag{1.5}$$

is called a *pLg-spline interpolating U.*

The terminology pLg-spline is a natural concatenation of earlier terminology well entrenched in the literature. Lg-splines (the case $p = 2$) were studied intensively in Jerome and Schumaker [17]. p-Splines were introduced in Jerome and Schumaker [18].

The main purpose of this paper is to examine the structure of pLg-splines for rather general types of constraint sets, including, for example, linear functionals with support over large sets (such as local integrals) as well as continuous inequality constraints, etc. Characterizations of pLg-splines with inequality constraints at an infinite number of points were obtained in Mangasarian and Schumaker [21] in optimal control terms, but explicit structural results were obtained there only for some cases involving polynomial splines (the case $L = D^m$). Golomb [14] considered extension problems similar to those discussed in Section 9 here for the case where $L = D^m$ (cf. also [18, 19]). Structural characterization of pLg-splines for a finite number of point constraints was carried out in Jerome [16] (cf. Theorem 7.1 here).

There has been relatively little work on splines corresponding to constraints defined by nonpoint functionals. Two examples in the case $p = 2$ involving local integrals are discussed in Anselone and Laurent [1]. Similar polynomial splines involving matching of areas led to the development of histosplines; see Boneva *et al.* [2], Schoenberg [25], and deBoor [3]. The first systematic treatment of constraint sets involving linear functionals with nonpoint support was carried out in the dissertation of the second-named author [5], on which this paper is based.

We begin the paper with two sections summarizing results on existence, uniqueness, and abstract characterization. This is followed by a section including basic notation and some preliminary results. The main results of this paper can be found in Sections 5–7 where we discuss the piecewise structure, smoothness properties, and structural characterization of pLg-splines for some fairly general classes of constraint sets of interest. Extension problems are treated in Section 8, and examples can be found in Section 9. Finally, we close the paper with a section including remarks and references to earlier work on pLg-splines.

2. Existence and Uniqueness

The following existence theorem covers most cases of interest.

THEOREM 2.1. *Let $U \neq \varnothing$ be defined by* (1.4) *and let* any one *of the following conditions hold*:

$$A \text{ is finite}, \tag{2.1}$$

$$-\infty < \inf_{\alpha \in A} \underline{y}_\alpha \quad \text{and} \quad \sup_{\alpha \in A} \bar{y}_\alpha < \infty, \tag{2.2}$$

$$N_L \cap \{f \in L_p{}^m[a, b]: \lambda_\alpha f \geqslant 0, \alpha \in A\} \subset U_0 = \{f \in L_p{}^m[a, b]: \lambda_\alpha f = 0, \alpha \in A\}, \tag{2.3}$$

where $N_L = \{f \in L_p{}^m[a, b]: \| Lf \|_p = 0\}$. Then there exists at least one pLg-spline interpolating U.

Proof. This theorem follows from general results in Daniel and Schumaker [6]. In particular, we note that L_p is reflexive for $1 < p < \infty$, while N_L is finite dimensional. ∎

Concerning uniqueness, we have by standard arguments:

THEOREM 2.2. *Any two solutions of* (1.5) *differ by an element in N_L. A necessary and sufficient condition for a function $s \in U$ to be the unique solution is that*

$$N_L \cap (U - s) = \{0\}. \tag{2.4}$$

In particular, a necessary condition for uniqueness to hold is that

$$N_L \cap U_0 = \{0\}. \tag{2.5}$$

When $\underline{y}_\alpha = \bar{y}_\alpha$ for all $\alpha \in A$, condition (2.5) *is also sufficient.*

3. Abstract Characterization

The structural and characterization results for pLg-splines in this paper are all based on the following quasi-orthogonality characterization of solutions of (1.5).

THEOREM 3.1. *A function $s \in U$ is a solution of* (1.5) *if and only if*

$$\int_a^b | Ls |^{p-1} \operatorname{sgn}(Ls)(Lu - Ls)(x)\, dx \geqslant 0, \qquad \text{all } u \in U. \tag{3.1}$$

Proof. There are several possible proofs of this theorem. For a proof based on Gateaux derivatives and results on convex minimization problems, see Jerome [16]. It can also be established by observing that

$$[f, g]_p = \int_a^b \frac{| g(x)|^{p-1} \operatorname{sgn}(g)(x) f(x)\, dx}{\|f\|_p^{p-2}}$$

constitutes a continuous semi-inner-product on $L_p[a, b]$ (cf. Giles [12]) and by generalizing the characterization of best approximation from linear subspaces in semi-inner-product spaces in Giles to a result on approximation from convex subsets. This result can also be proved directly using the Hahn–Banach and Riesz Representation Theorems, see Copley [5]. ∎

When U is a flat (i.e., a translate of a linear subspace), it is easily seen that the quasi-orthogonality condition becomes an orthogonality condition.

COROLLARY 3.2. *Suppose that $U = u + U_0$ for some $u \in L_p{}^m[a, b]$. Then a function $s \in U$ is a solution of* (1.5) *if and only if*

$$\int_a^b | Ls |^{p-1} \operatorname{sgn}(Ls)\, Lg(x)\, dx = 0, \qquad \text{all } g \in U_0. \tag{3.2}$$

4. PRELIMINARIES

The main purpose of this paper is to convert the abstract characterization of pLg-spline functions in Theorem 3.1 into precise structural properties. In this section we introduce some notation and develop some tools to assist in this task. Let $I = [a, b]$.

To begin with, we observe that the set of all bounded linear functionals on $L_p{}^m[I]$ is fairly large. It contains, for example, the class

$$\mathscr{L} = \left\{ \lambda\colon \lambda f = \sum_{j=0}^{m-1} \int_a^b f^{(j)}\, d\mu_j,\ \mu_j \text{ of bounded variation on } I \right\}. \tag{4.1}$$

The class $\mathscr{L}$ includes the so-called extended Hermite–Birkhoff (EHB) linear functionals of the form

$$\lambda f = \sum_{j=0}^{m-1} \gamma_j e_\xi^{(j)} f, \tag{4.2}$$

where $e_\xi^{(j)}$ is the point-evaluator functional defined by $e_\xi^{(j)} f = f^{(j)}(\xi)$.

As a tool in classifying linear functionals, it will be useful to introduce the notion of *support* of a linear functional. First, we recall that the support set of a function f is defined as the closure of $\{x \in I: f(x) \neq 0\}$. If λ is a linear functional defined on $L_p{}^m[I]$, we say that λ vanishes on an open set $0 \subseteq I$ provided $\lambda f = 0$ for all functions $f \in L_p{}^m[I]$ with support in 0. We define the *support of* λ as the complement of the largest open set on which λ vanishes. It follows that $\operatorname{supp}(\lambda)$ is a closed set, and that if $f \in L_p{}^m[I]$ is identically zero on an open set containing $\operatorname{supp}(\lambda)$, then $\lambda f = 0$.

We shall need several spaces of infinitely differentiable functions. We write $C^\infty(c, d)$ for the linear space of all infinitely differentiable functions on (c, d). It will be useful to introduce the following subspaces:

$$\underline{C}_j^\infty(c, d) = \{f \in C^\infty(c, d): f^{(i)}(c) = 0,\ i = 0, 1,..., j\}, \tag{4.3}$$

$$\overline{C}_j^\infty(c, d) = \{f \in C^\infty(c, d): f^{(i)}(d) = 0,\ i = 0, 1,..., j\}, \tag{4.4}$$

and

$$C_j^\infty(c, d) = \{f \in C^\infty(c, d): f^{(i)}(c) = f^{(i)}(d) = 0,\ i = 0,..., j\}. \tag{4.5}$$

We note that the lower (upper) bar means that the functions vanish at the lower (upper) endpoint up to the jth derivative.

In the following two lemmas we collect a number of facts concerning these spaces which will be of value later. These results follow from various facts about distributions, but for the reader's convenience, we give elementary direct proofs in Section 10.

LEMMA 4.1. *Suppose that $f \in L_p[c, d]$, some $1 < p < \infty$, and that*

$$\int_c^d f(x)\, \varphi(x)\, dx = 0 \qquad \textit{for all} \quad \varphi \in C^\infty(c, d). \tag{4.6}$$

Then $f(x) = 0$ for almost all $x \in (c, d)$. If for some $m > 0$,

$$\int_c^d f(x)\, \varphi^{(m)}(x)\, dx = 0 \qquad \textit{for all} \quad \varphi \in \underline{C}_{m-1}^\infty(c, d), \tag{4.7}$$

then $f(x) = 0$ a.e. on (c, d). The same result holds if the orthogonality in (4.7) is assumed for all $\varphi \in \overline{C}_{m=1}^\infty(c, d)$. Finally, if for some $m > 0$,

$$\int_c^d f(x)\, \varphi^{(m)}(x)\, dx = 0 \qquad \textit{for all} \quad \varphi \in C_{m-1}^\infty(c, d), \tag{4.8}$$

then there exists a polynomial $p_f \in \mathscr{P}_m = \{$space of polynomials of degree at most $m - 1\}$ such that $f = p_f$ a.e. on (c, d).

It will also be useful to have a version of this lemma for more general differential operators. Suppose M is a μth-order linear differential operator of the form

$$M = \sum_{j=0}^{\mu} b_j D^j, \quad b_\mu(x) \neq 0, \quad x \in I \qquad \text{and} \qquad b_j \in L_1{}^j[c, d], \quad j = 0,..., \mu. \tag{4.9}$$

The *formal adjoint* of M is defined by

$$M^*\varphi = \sum_{j=0}^{\mu} (-1)^j D^j(b_j\varphi). \tag{4.10}$$

LEMMA 4.2. *Suppose M is a μth-order linear differential operator as in* (4.9), *and suppose that $f \in L_p[I]$, some $1 < p < \infty$, is such that*

$$\int_c^d f(x)\, M\varphi(x)\, dx = 0 \qquad \textit{for all} \quad \varphi \in C_{\mu-1}^m(c, d). \tag{4.11}$$

Then there exists a function $\theta_f \in N_{M^*}$ such that $f = \theta_f$ a.e. on (c, d).

We conclude this section with the promised lemma on linear functionals with single-point support sets.

LEMMA 4.3. *Let λ be a bounded linear functional on $L_p{}^m[I]$ with support at a single point $\xi \in I$. Then $\lambda = \sum_0^{m-1} \gamma_i e_\xi^{(i)}$; i.e., λ is an EHB-linear functional.*

Proof. Let $\{\gamma_i\}_0^{m-1}$ be chosen so that $\tilde{\lambda} = \lambda - \sum_0^{m-1} \gamma_i e_\xi^{(i)}$ annihilates the space $\mathscr{P}_m$. Then by the definition of the Sobolev norm, if $p_f \in \mathscr{P}_m$ is chosen so that $(f - p_f)^{(j)}(a) = 0$, $j = 0, 1,..., m - 1$, then

$$\begin{aligned} |\tilde{\lambda} f| &= |\tilde{\lambda}(f - p_f)| \leqslant \|\tilde{\lambda}\| \, \|f - p_f\|_{L_p{}^m} \\ &= \|\tilde{\lambda}\| \cdot \|(f - p_f)^{(m)}\|_p = \|\tilde{\lambda}\| \, \|f^{(m)}\|_p . \end{aligned}$$

We conclude that $\tilde{\lambda}$ is bounded with respect to the semi-norm $\|f^{(m)}\|_p$ on $L_p{}^m[I]$. By the generalized Peano theorem of Sard [24], there exists $\psi_{\tilde{\lambda}} \in L_q{}^m[I]$, $1/p + 1/q = 1$, such that

$$\tilde{\lambda} f = \int_a^b \psi_{\tilde{\lambda}}(x) f^{(m)}(x)\, dx, \qquad \text{all } f \in L_p{}^m[I].$$

Since $\operatorname{supp}(\tilde{\lambda}) = \{\xi\}$, it follows that

$$0 = \tilde{\lambda}\varphi = \int_c^d \psi_{\tilde{\lambda}}(x)\, \varphi^{(m)}(x)\, dx, \qquad \text{all } \varphi \in \overline{C}_{m-1}^{\infty}(a, \xi) \cup \underline{C}_{m-1}^{\infty}(\xi, b).$$

Lemma 4.1 implies that $\psi_{\tilde{\lambda}} = 0$ a.e. on $(a, \xi) \cup (\xi, b)$, and we conclude that $\tilde{\lambda} = 0$, proving the lemma. ∎

5. Piecewise Structure

In this section we establish several theorems useful for determining the structure of pLg-splines on subintervals of I. The goal is to be able to partition I into subintervals so that the specific form of each component piece of the spline on each subinterval can be determined.

Our first result identifies the behavior of a pLg-spline s on an open interval J where none of the linear functionals in Λ have support, or if some do, the constraints are not active. It will be useful to introduce the notation

$$A(J) = \{\alpha \in A: \operatorname{supp}(\lambda_\alpha) \cap J \neq \varnothing\}. \tag{5.1}$$

This is the set of indices of linear functionals with support intersecting J.

Theorem 5.1 *Let J be an open subinterval of I. Suppose s is a solution of* (1.5) *such that for some $\delta > 0$,*

$$\lambda_\alpha s - \underline{y}_\alpha \geqslant \delta \qquad \text{and} \qquad \bar{y}_\alpha - \lambda_\alpha s \geqslant \delta, \qquad \text{all } \alpha \in A(J). \tag{5.2}$$

Then there exists $\theta_J \in N_{L^*}$ such that

$$| Ls |^{p-1} \operatorname{sgn}(Ls) = \theta_J \qquad \text{a.e. on } J. \tag{5.3}$$

Proof. Given $\varphi \in C_{m-1}^\infty(J)$, define

$$\begin{aligned} g(x) &= \delta\varphi(x)/C \,\|\, \phi \,\|_{L_p{}^m}, \qquad && x \in J \\ &= 0, && \text{otherwise}, \end{aligned} \tag{5.4}$$

where C is the constant in (1.3) which bounds the norms of all $\lambda \in \Lambda$. We claim $u = s + g \in U$. Indeed, $\lambda_\alpha u = \lambda_\alpha s$ for all $\alpha \in A \backslash A(J)$, while $\lambda_\alpha u \leqslant \lambda_\alpha s + \delta \leqslant \bar{y}_\alpha$ and $\lambda_\alpha u \geqslant \lambda_\alpha s - \delta \geqslant \underline{y}_\alpha$, all $\alpha \in A(J)$. By Theorem 3.1, $\int_J | Ls |^{p-1} \operatorname{sgn}(Ls)\, L\varphi \geqslant 0$. Since $u = s - g$ also belongs to U, we conclude this integral is actually 0, and since φ was arbitrary in $C_{m-1}^\infty(J)$, Lemma 4.2 with $M = L$ yields the result. ∎

The determination of the structure of pLg-splines on intervals where active constraints have support is more delicate. To identify the sets of linear functionals for which a specific spline s involves active constraints corresponding to linear functionals with support on an interval J, we define

$$\bar{A}(J, s) = \{\alpha \in A(J): \bar{y}_\alpha = \lambda_\alpha s\} \tag{5.5}$$

$$\underline{A}(J, s) = \{\alpha \in A(J): \underline{y}_\alpha = \lambda_\alpha s\}. \tag{5.6}$$

The set $\bar{A}(J, s)$ is the subset of those linear functionals with support on J such that the upper constraint is active, while $\underline{A}(J, s)$ is the subset where the lower is active. Either or both can be empty. Our next theorem has several applications involving linear functionals with active constraints.

THEOREM 5.2. *Let J be an open subinterval of I. Suppose there exists a positive δ such that*

$$\bar{y}_\alpha - \lambda_\alpha s \geqslant \delta \quad \text{and} \quad \lambda_\alpha s - \underline{y}_\alpha \geqslant \delta, \quad \text{all } \alpha \in A(J)\backslash(\underline{A}(J, s) \cup \bar{A}(J, s)). \tag{5.7}$$

In addition, suppose that there exists a bounded linear differential operator Q such that $M = LQ$ is a linear differential operator as in (4.9) *with*

$$\lambda_\alpha Q\varphi = 0 \quad \text{all} \quad \alpha \in \underline{A}(J, s) \cup \bar{A}(J, s) \quad \text{and all} \quad \varphi \in C_{\mu-1}^{\infty}(J),$$

where μ is the order of M. Then there exists $\theta_J \in N_{M^*}$ such that

$$| Ls |^{p-1} \operatorname{sgn}(Ls) = \theta_J \quad \text{a.e. on } J. \tag{5.8}$$

Proof. For any $\varphi \in C_{\mu-1}^{\infty}(J)$, let $g(x) = \delta Q\, \varphi(x)/C \parallel Q \parallel \cdot \parallel \varphi \parallel$ for $x \in J$ and $g(x) = 0$ otherwise. Then both $s + g$ and $s - g$ belong to U, and arguing as in the proof of Theorem 5.1, the result follows. ∎

To illustrate how this theorem might be applied, we consider the following specific corollary.

COROLLARY 5.3. *Let J be an open subinterval of I. Suppose that $\underline{A}(J, s) \cup \bar{A}(J, s) = \{\lambda^*\}$, where λ^* is a functional of the form*

$$\lambda^* f = \int_J f(x)\, w(x)\, dx, \qquad w \in C^m(J), \quad w(x) > 0 \quad \text{on } J. \tag{5.9}$$

Suppose that (5.7) *holds. Then there exists a function $\theta_J \in L_p{}^m(J)$ with*

$$| Ls |^{p-1} \operatorname{sgn}(Ls) = \theta_J \quad \text{a.e. on } J \tag{5.10}$$

and

$$L^*\theta_J = Kw(x), \qquad x \in J, \tag{5.11}$$

where

$$K \geqslant 0 \quad \text{if} \quad \lambda^* s < \bar{y}^* \quad \text{and} \quad K \leqslant 0 \quad \text{if} \quad \lambda^* s > \underline{y}^*. \tag{5.12}$$

Proof. We take $Q = (1/w)\, D$ in Theorem 5.2. For any $\varphi \in C_m{}^\infty(J)$, it is clear that $\lambda^* Q\varphi = \int_J \varphi' = 0$, so by the theorem, $| Ls |^{p-1} \operatorname{sgn}(Ls) = \theta_J$ a.e. where $\theta_j \in N_{M^*}$. Since $Q^* = -D(1/w)$, we find $L^*\theta_J = Kw$. The assertion about the sign of the constant K follows from Theorem 5.4 below. ∎

In addition to its use in the theorem above, the following result is also useful if, for example, we are constraining a spline to lie between two functions and we want to decide whether or not it can follow either the upper or lower boundary.

THEOREM 5.4. *Let J be an open subinterval of I, and suppose that s is such that $\theta = |Ls|^{p-1}\operatorname{sgn}(Ls) \in L_1^m(J)$. Suppose (5.7) holds. Finally, suppose that $\bar{A}(J, s) = \varnothing$, and that for every $(c, d) \subset J$, there exists $\varphi \in C_{m-1}^{\infty}(J)$ with $\varphi(x) > 0$ on (c, d) and $\lambda_\alpha \varphi \geqslant 0$ for all $\alpha \in \underline{A}(J, s)$. Then*

$$L^*\theta \geqslant 0 \qquad \text{a.e. on } J. \tag{5.13}$$

If we assume instead that $\underline{A}(J, s) = \varnothing$ and that there exists φ as above with $\varphi(x) < 0$ on (c, d) and $\lambda_\alpha \varphi \leqslant 0$ all $\alpha \in \bar{A}(J, s)$, then

$$L^*\theta \leqslant 0 \qquad \text{a.e. on } J. \tag{5.14}$$

Proof. We consider the case where $\bar{A}(J, s) = \varnothing$; the other case is similar. Suppose that $L^*\theta < 0$ on a subset D of J with $\sigma(D) = \rho > 0$, where σ stands for Lebesgue measure. We know (cf. Royden [23, p. 62]) that for each $0 < \epsilon < \rho/2$, there exists $0 = \bigcup_1^n I_i$, where I_i are disjoint open intervals with $\sigma((O \backslash D) \cup (D \backslash O)) < \epsilon$. Thus, $\sigma(O \cap D) > \rho/2$ and $\sigma(O^J \cap D) < \epsilon$, where O^J is the complement of O in J. Let $\varphi \in C_c^\infty(0)$ with $0 < \varphi(x) < 1$, and let g be defined as in (5.4). By hypothesis, we can choose φ so that $\lambda_\alpha \varphi \geqslant 0$ for all $\alpha \in \underline{A}(J, s)$. Then $s + g \in U$, and by Theorem 3.1,

$$0 \leqslant \int_D L^*\theta\varphi = \int_{O \cap D} L^*\theta\varphi + \int_{O^J \cap D} L^*\theta\varphi.$$

But the first integral on the right is negative, and the second can be made arbitrarily small if we take ϵ sufficiently small. This contradiction implies that $L^*\theta < 0$ on a subset of positive measure is impossible. ∎

Our last piecewise structural result concerns intervals near a and b where there are no active constraints.

THEOREM 5.5. *Let $\underline{x} = \inf_{\alpha \in A^*} \inf(\operatorname{supp} \lambda_\alpha)$ and $\bar{x} = \sup_{\alpha \in A^*} \sup(\operatorname{supp} \lambda_\alpha)$ where $A^* = \underline{A}(I, s) \cup \bar{A}(I, s)$. (Thus there are no active constraints involving linear functionals with support to the left of $\underline{x}$ or to the right of $\bar{x}$.) Suppose that for all $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ with*

$$\bar{y}_\alpha - \lambda_\alpha s \geqslant \delta \qquad \text{and} \qquad \lambda_\alpha s - \underline{y}_\alpha \geqslant \delta \quad \text{for all } \alpha \in A([\bar{x} + \epsilon, b]). \tag{5.15}$$

Then

$$Ls = 0 \qquad \text{a.e. on } (\bar{x}, b]. \tag{5.16}$$

If (5.15) *holds for all* $\alpha \in A([a, \underline{x} - \epsilon])$, *then*

$$Ls = 0 \qquad \text{a.e. on } [a, \underline{x}). \tag{5.17}$$

Proof. Suppose $Ls \neq 0$ a.e. on $(\bar{x}, b]$. Then, for all c sufficiently near $\bar{x}$ with $\bar{x} < c < b$, $\int_c^b | Ls |^p > 0$. Choose $u \in N_L$ with $u^{(j)}(c) = s^{(j)}(c)$, $j = 0, 1, 2,..., m - 1$. Set

$$\begin{aligned} \tilde{s} &= s, \qquad a \leqslant x \leqslant c \\ &= u, \qquad c \leqslant x \leqslant b. \end{aligned}$$

For each $0 < \beta < 1$, let $\hat{s} = \beta\tilde{s} + (1 - \beta)\, s$. We claim that for β sufficiently small, $\hat{s} \in U$. Indeed, with $\epsilon = (c - \bar{x})/2$, we note that for all $\alpha \in A([a, \bar{x} + \epsilon])$, $\lambda_\alpha \hat{s} = \lambda_\alpha s$. On the other hand, for $\alpha \in A([\bar{x} + \epsilon, b])$, we have

$$\underline{y}_\alpha + \delta(\epsilon) \leqslant \lambda_\alpha s \leqslant \bar{y}_\alpha - \delta(\epsilon).$$

But $\lambda_\alpha \hat{s} = (1 - \beta)\, \lambda_\alpha s + \beta \lambda_\alpha \tilde{s}$, and $| \lambda_\alpha \tilde{s} | \leqslant \| \lambda_\alpha \| \| \tilde{s} \| < C \| \tilde{s} \|$. Thus, for β sufficiently small, $\hat{s}$ is indeed in U.

Now

$$\| L\hat{s} \|_p^p = \| L(\beta\tilde{s} + (1 - \beta)\, s)\|_p^p = \int_a^c | Ls |^p + \int_c^b | L(\beta\tilde{s} + (1 - \beta)\, s)|^p$$

$$= \int_a^c | Ls |^p + (1 - \beta)^p \int_c^b | Ls |^p < \| Ls \|_p^p .$$

Thus $\hat{s} \in U$ is better than s, and this contradiction implies $Ls = 0$ a.e. on $[c, b]$. As c was arbitrary, (5.16) follows. The proof of (5.17) is similar. ∎

6. Smoothness Properties

In this section we continue our program of developing tools which will help us to delineate the structure of pLg-splines. Using the results of Section 5, we can often determine the form of s on open subintervals of I. Here, we want to examine the manner in which two component pieces of s defined on adjoining subintervals of I tie together at the common endpoint. Such a join point is called a *knot* of the spline. Generally, it is difficult to identify which points of I will be knots. We shall see, however, that for several classes of linear functionals, knot points can be singled out, and moreover, the behavior of the spline there can be described (usually in terms of the jump in s or of certain linear combinations of s and its derivatives).

In order to state these jump conditions, we need to introduce the linear differential operators

$$L_i = \sum_{j=0}^{m-i-1} a_{j+i+1} D^j, \qquad i = -1, 0,..., m - 1 \tag{6.1}$$

and their formal adjoints

$$L_i^* = \sum_{j=0}^{m-i-1} (-1)^j D^j[a_{i+j+1}]. \tag{6.2}$$

The operators $L_0^*,\ldots, L_{m-1}^*$ are called the partial adjoints of L. L itself corresponds to L_{-1} in this notation. Finally, we define

$$\begin{aligned} \operatorname{jump}[\varphi]_z &= \varphi(z+) - \varphi(z-), & a < z < b,\\ &= -\varphi(b-), & z = b,\\ &= \varphi(a+), & z = a. \end{aligned} \tag{6.3}$$

Our first smoothness theorem concerns the case where there are linear functionals with support at a point ξ, while the other linear functionals in Λ with support in a neighborhood of ξ are inactive. In this case ξ is always a knot, and the behavior of s at ξ is described in the following theorem.

THEOREM 6.1. *Let s be a solution of* (1.5), *and suppose that for some $a \leqslant \xi \leqslant b$, the set Λ defining U includes the EHB-functionals*

$$\lambda_{(\xi,i)} = \sum_{j=0}^{m-1} \gamma_{ij}(\xi)\, e_\xi^{(j)}, \qquad i = 0, 1,\ldots, l(\xi) - 1, \tag{6.4}$$

for some $l \leqslant m$. Suppose that for some $\epsilon > 0$ and $\delta > 0$, all of the other linear functionals $\lambda_\alpha \in \Lambda$ with $\operatorname{supp}(\lambda_\alpha) \cap (\xi - \epsilon, \xi + \epsilon) \neq \varnothing$ *satisfy*

$$\bar{y}_\alpha - \lambda_\alpha s \geqslant \delta \qquad \textit{and} \qquad \lambda_\alpha s - \underline{y}_\alpha \geqslant \delta. \tag{6.5}$$

Suppose the set in (6.4) *is linearly independent, or what is equivalent, that the matrix $\gamma = (\gamma_{ij})_{i=0,j=0}^{l-1,m-1}$ is of full rank l. Let $\tilde{\gamma}$ be any nonsingular augmentation of γ, and let η be the inverse of $\tilde{\gamma}^T$. Define*

$$Q_{(\xi,i)} = \sum_{j=0}^{m-1} \eta_{ij} L_j^*, \qquad i = 0, 1,\ldots, m - 1, \tag{6.6}$$

where the L_j^* are the partial adjoints in (6.2), *and set*

$$R_{(\xi,i)}s = -Q_{(\xi,i)}(|\, Ls\,|^{p-1} \operatorname{sgn}(Ls)), \qquad i = 0, 1,\ldots, m - 1. \tag{6.7}$$

Then

$$\operatorname{jump}[R_{(\xi,i)}s]_\xi = 0, \qquad i = l,\ldots, m - 1. \tag{6.8}$$

Moreover, for $i = 0, 1,\ldots, l - 1$,

$$\operatorname{jump}[R_{(\xi,i)}s]_\xi \leqslant 0 \qquad \textit{if} \quad \lambda_{(\xi,i)}s > \underline{y}_{(\xi,i)}\,, \tag{6.9}$$

$$\operatorname{jump}[R_{(\xi,i)}s]_\xi \geqslant 0 \qquad \textit{if} \quad \lambda_{(\xi,i)}s < \bar{y}_{(\xi,i)}\,. \tag{6.10}$$

In particular,

$$\text{jump}[R_{(\xi,i)}s]_\xi = 0 \qquad \text{if} \quad \underset{\sim}{y}_{(\xi,i)} < \lambda_{(\xi,i)}s < \bar{y}_{(\xi,i)} . \tag{6.11}$$

Proof. Since this proof follows closely similar ones in less general cases (cf. [14, 16, 17]), we shall be brief. Suppose $a < \xi < b$, and that j is fixed with $0 \leqslant j \leqslant m - 1$. Let P be a polynomial of degree $m - 1$ such that

$$\tilde{\lambda}_{(\xi,i)}P = \sum_{v=0}^{m-1} \tilde{\gamma}_{iv}P^{(v)}(\xi) = \delta_{ij}d_i , \tag{6.12}$$

where

$$\begin{aligned} d_i &= \bar{y}_{(\xi,i)} - \lambda_{(\xi,i)}s, && \text{if} \quad \bar{y}_{(\xi,i)} > \lambda_{(\xi,i)}s, \\ &= \underset{\sim}{y}_{(\xi,i)} - \lambda_{(\xi,i)}s, && \text{otherwise, } i = 0, 1,..., l - 1, \\ &= 1, && \qquad\qquad i = l,..., m - 1. \end{aligned} \tag{6.13}$$

Given $\psi \in C_{m-1}^\infty(J)$, with $0 \leqslant \psi \leqslant 1$ and $\psi(x) \equiv 1$ on $(\xi - (\epsilon/2), \xi + (\epsilon/2))$, let $\varphi = \psi \cdot P$. By Leibnitz's rule, we also note that $\tilde{\lambda}_{(\xi,i)}\varphi = \delta_{ij}d_i$, $i = 0, 1,..., m - 1$. Then, with g defined as in (5.4), it is clear that $s + g \in U$. From Theorem 3.1 we conclude that

$$0 \leqslant \int_a^b \theta Lg = \int_{\xi-\epsilon}^{\xi+\epsilon} \theta Lg,$$

where $\theta = | Ls |^{p-1} \operatorname{sgn}(Ls)$. Integrating by parts, using Theorem 5.1, and taking account of the relation between the L^*'s and R's (cf. [17]), we obtain

$$\begin{aligned} 0 &\leqslant \int_{\xi-\epsilon}^{\xi} \varphi L^*\theta + \int_{\xi}^{\xi+\epsilon} \varphi L^*\theta - \sum_{i=0}^{m-1} \varphi^{(i)} \text{jump}[L_i^*\theta]_\xi \\ &= \sum_{i=0}^{m-1} \tilde{\lambda}_{(\xi,i)}\varphi \, \text{jump}[R_{(\xi,i)}s]_\xi = d_j \, \text{jump}[R_{(\xi,j)}s]_\xi . \end{aligned}$$

If $0 \leqslant j \leqslant l - 1$, this implies (6.9)–(6.10). The condition (6.11) follows by combining these two. Now if $l \leqslant j \leqslant m - 1$, then we notice that $s + g \in U$ if we replace $d_j = 1$ by $d_j = -1$, and we conclude as above that $0 = d_j \, \text{jump}[R_{(\xi,j)}s]_\xi$, and thus that $0 = \text{jump}[R_{(\xi,j)}s]_\xi$, which is (6.8). When $s = a$ or $\xi = b$, the proof is similar. ∎

The minus sign introduced in the definition of the R operators in (6.7) has been introduced so that R agrees with operators previously used in the literature; e.g., see [17].

As would be expected, identifying a point ξ as a knot when there are active linear functionals with support in a neighborhood of ξ (and finding the corresponding behavior of s there) is considerably more difficult, in general, than the cases considered above. We can, however, obtain some results for several cases of interest.

THEOREM 6.2. *Let $a < \xi < b$, and suppose Λ contains a set of linear functionals of the form* (6.4) *with support at ξ (we allow $l(\xi) = 0$) as well as two linear functionals of the form*

$$\lambda_1\varphi = \int_{\xi_1}^{\xi} \varphi w_1, \qquad \lambda_2\varphi = \int_{\xi}^{\xi_2} w_2\varphi, \qquad a < \xi_1 < \xi < \xi_2 < b \tag{6.14}$$

with w_1 and w_2 positive functions in $C^m(\xi_1, \xi)$ and $C^m(\xi, \xi_2)$, respectively.

Suppose for some $\epsilon > 0$ and $\delta(\epsilon) > 0$, all of the other linear functionals $\lambda \in \Lambda$ with $\operatorname{supp}(\lambda) \cap (\xi - 2\epsilon, \xi + 2\epsilon) \neq \varnothing$ *satisfy* (6.5). *Then the assertions* (6.8)–(6.11) *of Theorem* 6.1 *hold. The same result holds if Λ contains just one of the λ_1, λ_2, but not the other.*

Proof. The proof is similar to the proof of Theorem 6.1. For example, to prove (6.8), let $\varphi_1 \in C_{m-1}^{\infty}(\xi - \epsilon, \xi + \epsilon)$ be constructed exactly as in the proof of Theorem 6.1. Let $\varphi_2 \in C_{m-1}^{\infty}(\xi - 2\epsilon, \xi - \epsilon)$ and $\varphi_3 \in C_{m-1}^{\infty}(\xi + \epsilon, \xi + 2\epsilon)$ be such that

$$\int_{\xi-2\epsilon}^{\xi-\epsilon} w_1\varphi_2 = -\int_{\xi-\epsilon}^{\xi} w_1\varphi_1 \qquad \text{and} \qquad \int_{\xi+\epsilon}^{\xi+2\epsilon} w_2\varphi_3 = -\int_{\xi}^{\xi+\epsilon} w_2\varphi_1 .$$

Now define

$$\begin{aligned} \varphi(x) &= \varphi_1(x), & & x \in (\xi - \epsilon, \xi + \epsilon) \cap I, \\ &= \varphi_2(x), & & x \in (\xi - 2\epsilon, \xi - \epsilon) \cap I, \\ &= \varphi_3(x), & & x \in (\xi + \epsilon, \xi + 2\epsilon) \cap I, \\ &= 0, & & \text{otherwise.} \end{aligned}$$

With g defined as in (5.4), we check easilily that $s \pm g$ both belong to U, and that $\lambda_{(\xi,j)}\, g = 1$. Using Theorem 3.1 and integration by parts just as in the proof of Theorem 6.1, we obtain (with $\theta = |\, Ls\,|^{p-1} \operatorname{sgn}(Ls)$)

$$0 = \int_a^b \theta Lg = \int_{\xi-\epsilon}^{\xi+\epsilon} L^*\theta g + \operatorname{jump}[R_{(\xi,j)}s]_\xi .$$

The result would follow if we did not have the extra integral. We may get rid of it as follows. Let ϵ_ν be a sequence converging to 0, and for each ν, let g_ν be constructed as above. Then, since $L^*\theta = K_1 w_1$ on $(\xi - \epsilon, \xi)$ and $K_2 w_2$ on $(\xi, \xi + \epsilon)$, as $\nu \to \infty$ the integral approaches 0. The condition (6.8) follows. The proofs of the other assertions follow in the same way. If ξ is one of the endpoints a or b and Λ contains an integral type linear functional on one side of it, a similar analysis is valid. ∎

Our next result concerns another case where certain linear functionals are active in every neighborhood of ξ.

THEOREM 6.3. *Let $a \leqslant \xi \leqslant b$, and suppose Λ contains a set of linear functionals of the form* (6.4) *with support at ξ, where the first of these is point evaluation, $\lambda_{(\xi,0)} = e_\xi$. Suppose that, in addition, Λ contains the linear functional $\lambda_{(t,0)} = e_t$ for all $t \in J = (\xi - \epsilon, \xi + \epsilon) \cap I$. Suppose that for some $\delta > 0$, all other linear functionals in Λ with support intersecting J satisfy* (6.5), *and that $\bar{y}_{(t,0)} - y_{(t,0)} \geqslant \delta$ for all $t \in J$. Let s be a solution of* (1.5), *and suppose that $\theta = | Ls |^{p-1} \operatorname{sgn}(Ls)$ is such that $L^*\theta$ is integrable on J. Then the smoothness conditions* (6.8)–(6.11) *hold.*

Proof. The proof is similar to that of Theorem 6.1. Let $d_0, \ldots, d_{m-1}$ be as defined in (6.13), and let $0 \leqslant \eta_i \leqslant 1$, $i = 0, 1, \ldots, m-1$ be prescribed. Let P be a polynomial of degree $m - 1$ such that

$$\tilde{\lambda}_{(\xi,i)} P = \eta_i d_i, \qquad i = 0, 1, \ldots, m-1.$$

For each positive integer n, let $\psi_n \in C_{m-1}^{\infty}(\xi - \epsilon/n, \xi + \epsilon/n)$ be such that $0 \leqslant \psi_n \leqslant 1$ and $\psi_n(x) = 1$ on $(\xi - \epsilon/2n, \xi + \epsilon/2n)$. Then as before, $\varphi_n = \psi_n P$ satisfies $\tilde{\lambda}_{(\xi,i)} \varphi_n = \eta_i d_i$, $i = 0, 1, \ldots, m-1$. Moreover, since P is continuous, if $\eta_0 \neq 0$ and n is sufficiently large, then $d_0 P(x) > 0$ for all $x \in (\xi - \epsilon/n, \xi + \epsilon/n)$. Integrating by parts as before leads to

$$0 \leqslant \int_{\xi-\epsilon/n}^{\xi} \varphi_n L^*\theta + \int_{\xi}^{\xi+\epsilon/n} \varphi_n L^*\theta + \sum_{i=0}^{m-1} \eta_i d_i \operatorname{jump}[R_{(\xi,i)} s]_\xi .$$

Since $L^*\theta$ is integrable, as $n \to \infty$ the integrals vanish, and we conclude that

$$0 \leqslant \sum_{i=0}^{m-1} \eta_i d_i \operatorname{jump}[R_{(\xi,i)}]_\xi .$$

Now, choosing $\eta_0 = 1$ and the remaining η's to be zero, we obtain (6.9)–(6.10) for $j = 0$. Fixing $1 \leqslant j \leqslant m - 1$, we may next take all the η's equal to 0 except for η_0 and η_j. By making the ratio η_0/η_j arbitrarily close to 0, (6.9)–(6.10) follows for general j. Finally, we observe that for $l \leqslant j \leqslant m - 1$, we may switch η_j to a negative number, which implies (6.8). ∎

We note that it was necessary in Theorem 6.3 to assume the integrability of $L^*(| Ls |^{p-1} \operatorname{sgn}(Ls))$ near ξ, as it does not follow from any piecewise structural properties. In the special case where we are restricting a spline by forcing it to lie between two given curves, the smoothness conditions (6.8)–(6.11) can be useful in deciding when the spline can get on or off one of the boundary curves, and are thus useful in helping locate possible knots.

We should also remark that the assumption $\bar{y}_{(t,i)} - \underline{y}_{(t,i)} \geqslant \delta > 0$ is critical to the proof of Theorem 6.3. If this condition is not assumed, it becomes highly problematical to choose φ_n such that $s + g_n$ remains in U. Moreover, if the upper and lower constraints are allowed to become arbitrarily close (or touch), this often induces *implied constraints*. To illustrate this, suppose that $s(t)$ is required to lie between the functions $\bar{y}(t) = t^2$ and $\underline{y}(t) = 0$ for t in a neighborhood of 0. Then at 0 the function s would have to satisfy $s(0) = s'(0) = 0$. If the upper constraint were defined by $\bar{y}(t) = t^4$, the $s''(0) = s^{(3)}(0) = 0$ would also be automatically forced. The following lemma illustrates this more vividly.

LEMMA 6.4. *Let t_ν be an increasing sequence converging to $\xi \in I$. Suppose that $g \in L_p{}^m[t_0, \xi]$ is such that $g(t_\nu) = 0$ for all ν. Then*

$$D^j g(t_{N(\nu)}) = o(\delta_\nu^{m-j-1+1/q}), \qquad j = 1, 2,..., m - 1, \tag{6.15}$$

where $1/p + 1/q = 1$, and $N(\nu)$ and δ_ν are such that

$$t_{N(\nu)+1} - t_{N(\nu)} = \delta_\nu = \max_{1 \leqslant i \leqslant m-1} (t_{\nu+i+1} - t_{\nu+i}).$$

In particular, $D^j g(\xi) = 0, j = 0, 1,..., m - 1$ is forced.

Proof. See the proof of Theorem 5.1 in Golomb [14]. ∎

7. STRUCTURAL CHARACTERIZATION OF pLg-SPLINES

The quasi-orthogonality condition of Theorem 3.1 is a necessary and sufficient condition for a function s to be a solution of problem (1.5); that is, it completely characterizes pLg-splines. In Sections 5 and 6 we have used this condition to obtain specific structural results for pLg-splines. In this section we show that in many cases we can identify a sufficient amount of structural information to characterize the spline.

We begin with the case where the support of the set of linear functionals Λ defining U as in (1.4) is a finite set of isolated points. The following theorem recovers a result of Jerome [16].

THEOREM 7.1. *Let $a = x_0 \leqslant x_1 < \cdots < x_k \leqslant x_{k+1} = b$, and suppose*

$$\Lambda = \left\{ \lambda_{(x_i,j)} = \sum_{\nu=0}^{m-1} \gamma_{j\nu}(x_i)\, e_{x_i}^{(\nu)} \right\}_{i=1}^{k}\,{}_{j=0}^{l(x_i)-1}.$$

Suppose for each $i = 1, 2,..., k$ that the matrix $(\gamma_{j\nu})$ is of full rank $l(x_i)$ (i.e., the EHB-linear functionals associated with each x_i are linearly independent). Let $\underline{y}_{(x_i,j)} \leqslant \bar{y}_{(x_i,j)}$, $j = 0, 1,..., l(x_i) - 1$, $i = 1, 2,..., k$ be prescribed real *numbers, and let* U *be defined by* (1.4). *If* s *is a pLg-spline interpolating* U, *then it must satisfy the following conditions*:

For each $j = 1, 2,..., k-1$, *there exists* $\theta_j \in N_{L^*}$ *such that*

$$|Ls|^{p-1}\operatorname{sgn}(Ls) = \theta_j \qquad a.e.\ on\ (x_j, x_{j+1}); \tag{7.1}$$

$$Ls = 0 \qquad a.e.\ on\ (a, x_1)\ and\ on\ (x_k, b); \tag{7.2}$$

$$\operatorname{jump}[R_{(x_i,j)}s]_{x_i} = 0, \qquad j = l(x_i),..., m-1, \quad i = 1, 2,..., k; \tag{7.3}$$

$$\operatorname{jump}[R_{(x_i,j)}s]_{x_i} \leqslant 0 \qquad if \quad \lambda_{(x_i,j)}s > \underline{y}_{(x_i,j)}\,; \tag{7.4}$$

and

$$\operatorname{jump}[R_{(x_i,j)}s]_{x_i} \geqslant 0 \qquad if \quad \lambda_{(x_i,j)}s < \bar{y}_{(x_i,j)} \tag{7.5}$$

for $j = 0, 1,..., l(x_i) - 1$ *and* $i = 1, 2,..., k$. *Conversely, if* $s \in U$ *satisfies conditions* (7.1)–(7.5), *then it is a pLg-spline interpolating* U; *i.e., a solution of* (1.5). *In particular, if* U *is such that there is a unique pLg-spline interpolating it, then the spline is completely characterized by the properties* (7.1)–(7.5) *as an element of* U.

Proof. The necessity of (7.1) was proved in Theorem 5.1, while the necessity of (7.2) was the content of Theorem 5.5. The smoothness assertions (7.3)–(7.5) are contained in Theorem 6.1.

We turn now to the converse. Suppose $s \in U$ satisfies (7.1)–(7.5). To prove s solves (1.5), it suffices by Theorem 3.1 to verify (3.1). To this end, let $u \in U$, and set $g = u - s$. Then, with $\theta = |Ls|^{p-1}\operatorname{sgn}(Ls)$, integration by parts and the use of (7.2) leads to

$$\int_a^b \theta Lg = \int_{x_1}^{x_k} \theta Lg = \sum_{i=1}^{k-1} \int_{x_i}^{x_{i+1}} \theta Lg$$

$$= \sum_{i=1}^{k-1} \int_{x_i}^{x_{i+1}} gL^*\theta - \sum_{i=1}^{k} \sum_{j=0}^{m-1} g^{(j)}(x_i)\operatorname{jump}[L_j^*\theta]_{x_i}.$$

The integrals vanish in view of (7.1). With $\tilde{\lambda}$'s as in (6.12), the sum is equal to (cf. the proof of Theorem 6.1)

$$\sum_{i=1}^{k} \sum_{j=0}^{m-1} \tilde{\lambda}_{(x_i,j)}g\operatorname{jump}[R_{(x_i,j)}s]_{x_i}.$$

For each $i = 1, 2,..., k$, the terms in this sum with $l(x_i) \leqslant j \leqslant m-1$ are 0 by (7.3). For $0 \leqslant j \leqslant l(x_i) - 1$, $\operatorname{jump}[R_{(x_i,j)}s]_{x_i}$ and $\tilde{\lambda}_{(x_i,j)}g$ have the same signs by (7.4)–(7.5); (note that the $\tilde{\lambda}_j$ and λ_j are the same for these j). We conclude that $\int_a^b Lg\theta \geqslant 0$, which is (3.1). By Theorem 3.1, s is a pLg-spline interpolating U. ∎

Our next two characterization theorems are concerned with constraint sets which include linear functionals with support over intervals. The first theorem deals with a constraint on the spline to lie between two prescribed functions throughout I, coupled with some EHB constraints at a finite number of isolated points.

THEOREM 7.2. *Let $a = x_0 \leqslant x_1 < \cdots < x_k \leqslant x_{k+1} = b$, and suppose Λ contains the linear functionals $\lambda_{(t,0)} = e_t$, all $a \leqslant t \leqslant b$, as well as EHB-functionals*

$$\lambda_{(x_i,j)} = \sum_{\nu=0}^{m-1} \gamma_{j\nu}(x_i)\, e_{x_i}^{(\nu)}, \qquad j = 1, 2,..., l(x_i) - 1 \quad \text{and} \quad i = 1, 2,..., k.$$

Suppose $\underline{y}_{(x_i,j)} \leqslant \bar{y}_{(x_i,j)}$ are prescribed real numbers for $j = 1, 2,..., l(x_i) - 1$ and $i = 1, 2,..., k$ and that $\underline{y}_{(t,0)} < \bar{y}_{(t,0)}$ are prescribed functions in $L_p^{2m}[I]$. We assume the matrices $(\gamma_{j\nu}(x_i))_{j=0,\ i=1}^{l(x_i-1),\ k}$ are of full rank. Suppose s is a pLg-spline interpolating U, i.e., a solution of (1.5). We write $\theta = |\, Ls\,|^{p-1}\operatorname{sgn}(Ls)$. Then, there exists a finite set of points $\Delta = \{\tilde{a} = \tilde{x}_0 < \tilde{x}_1 < \cdots < \tilde{x}_n < \tilde{x}_{n+1} = b\}$ which includes the points $\{x_i\}_0^{k+1}$ and all points where s gets on or off the boundary. Moreover, if we set

$$\begin{aligned} E &= \{0, 1,..., n\}, \\ \bar{E} &= \{i \in E\colon s(t) = \bar{y}_{(t,0)}\,,\ t \in (\tilde{x}_i\,,\ \tilde{x}_{i+1})\}, \\ \underline{E} &= \{i \in E\colon s(t) = \underline{y}_{(t,0)}\,,\ t \in (\tilde{x}_i\,,\ \tilde{x}_{i+1})\}, \\ E_0 &= E \backslash (\bar{E} \cup \underline{E}), \end{aligned} \tag{7.6}$$

then s must also satisfy the following conditions:

There exist θ_i in N_{L^*} such that

$$|\, Ls\,|^{p-1}\operatorname{sgn}(Ls) = \theta_i \qquad \text{a.e. on } (\tilde{x}_i\,,\ \tilde{x}_{i+1}),\quad i \in E_0\,; \tag{7.7}$$

$$L^*\theta \leqslant 0 \qquad \text{a.e. on } (\tilde{x}_i\,,\ \tilde{x}_{i+1}),\quad i \in \bar{E}; \tag{7.8}$$

$$L^*\theta \geqslant 0 \qquad \text{a.e. on } (\tilde{x}_i\,,\ \tilde{x}_{i+1}),\quad i \in \underline{E}; \tag{7.9}$$

$$Ls = 0 \qquad \text{a.e. on } \begin{cases} (a, \tilde{x}_1) & \text{if } 0 \in E_0\,; \\ (\tilde{x}_n\,, b) & \text{if } n \in E_0\,; \end{cases} \tag{7.10}$$

$$\operatorname{jump}[R_{(x_i,j)}s]_{x_i} = 0, \qquad j = l(x_i),..., m-1, \quad \text{and} \quad i = 1, 2,..., k; \tag{7.11}$$

$$\operatorname{jump}[R_{(x_i,j)}s]_{x_i} \leqslant 0 \qquad \text{if} \quad \lambda_{(x_i,j)}s > \underline{y}_{(x_i,j)} \tag{7.12}$$

$$\operatorname{jump}[R_{(x_i,j)}s]_{x_i} \geqslant 0 \qquad \text{if} \quad \lambda_{(x_i,j)}s < \bar{y}_{(x_i,j)}\,, \quad j = 0, 1,..., l(x_i) - 1, \quad \text{and} \quad i = 1, 2,..., k; \tag{7.13}$$

$$\operatorname{jump}[R_{(t,0)}s]_t \leqslant 0 \qquad \text{if} \quad s(t) = \bar{y}_{(t,0)}\,, \quad t \in \Delta; \tag{7.14}$$

$$\operatorname{jump}[R_{(t,0)}s]_t \geqslant 0 \qquad \text{if} \quad s(t) = \underline{y}_{(t,0)}\,, \quad t \in \Delta. \tag{7.15}$$

Conversely, if $s \in U$ is a function such that for some points $\{\tilde{x}_i\}_0^{n+1}$, including the points $\{x_i\}_0^{k+1}$, conditions (7.7)–(7.15) *are satisfied, then s is a pLg-spline interpolating U.*

Proof. First we observe that a function $s \in U$ cannot jump between $\bar{y}_{(t,0)}$ and $\underline{y}_{(t,0)}$ more than a finite number of times. Indeed, since $s \in L_p{}^m[I]$ implies it is of bounded variation while $\bar{y}$ and $\underline{y}$ are bounded apart by $\delta > 0$, the assertion follows. The necessity of conditions (7.7), (7.8)–(7.9), and (7.10) follows from Theorems 5.1, 5.4, and 5.5, respectively. The smoothness conditions (7.11)–(7.15) follow from Theorem 6.3. (Conditions (7.14) and (7.15) are singled out because of the importance of the constraint on s.)

To prove the converse, we check (3.1). Suppose $u \in U$, and set $g = u - s$. We now have

$$\int_a^b \theta L g = \sum_{i=0}^{n} \int_{\tilde{x}_i}^{\tilde{x}_{i+1}} g L^* \theta - \sum_{i=0}^{n} \sum_{j=0}^{m} g^{(j)}(\tilde{x}_i) \operatorname{jump}[L_j{}^* \theta]_{\tilde{x}_i} .$$

Each of the integrals in the first sum is nonnegative, since, e.g., if $g(x) > 0$ and $\theta \neq 0$ on some subinterval $(\tilde{x}_i, \tilde{x}_{i+1})$, then $\underline{y}_{(x,0)} = s(x) < u(x) \leqslant \bar{y}_{(x,0)}$ on this interval, and by (7.9), $L^*\theta \geqslant 0$. The fact that the terms in the double sum are all nonpositive follows just as in the proof of Theorem 7.1. ∎

To illustrate once more how the tools of Sections 5 and 6 can be used to characterize pLg-splines with linear functionals with support over subintervals, we consider the case where integral functionals are involved.

THEOREM 7.3. *Let $a \leqslant \underline{x}_1 < \bar{x}^1 \leqslant \underline{x}_2 < \bar{x}_2 \leqslant \cdots \leqslant \bar{x}_k \leqslant b$. Suppose $w_i \in C^m[\underline{x}_i, \bar{x}_i]$ are positive functions on $(\underline{x}_i, \bar{x}_i)$, $i = 1, 2,..., k$. Let $\Lambda = \{\lambda_i : \lambda_i f = \int_{\underline{x}_i}^{\bar{x}_i} w_i(x) f(x)\, dx,\ i = 1,..., k\}$, and suppose U is defined as in* (1.4) *with prescribed $\underline{y}_i < \bar{y}_i$, $i = 1, 2,..., k$. Then any pLg-spline s interpolating U must satisfy the following conditions:*

There exist functions $\theta_i \in N_{L^*}$, $i = 1, 2,..., k - 1$ with

$$| Ls |^{p-1} \operatorname{sgn}(Ls) = \theta_i \qquad \text{a.e. on } (\bar{x}_i, \underline{x}_{i+1}); \tag{7.16}$$

There exists a function θ with $\theta = | Ls |^{p-1} \operatorname{sgn}(Ls)$ a.e. such that $L^*\theta = K_i w_i$ on (x_i, x_i), with (7.17)

$$\begin{matrix} K_i \geqslant 0 & \text{if } \lambda_i s < \bar{y}_i , \\ K_i \leqslant 0 & \text{if } \lambda_i s > \underline{y}_i , \end{matrix} \qquad i = 1, 2,..., k;$$

$$Ls = 0 \qquad \text{a.e. on } (a, \underline{x}_1) \text{ and on } (\bar{x}_k, b); \tag{7.18}$$

$$\operatorname{jump}[L_j{}^*\theta]_{\underline{x}_i} = \operatorname{jump}[L_j{}^*\theta]_{\bar{x}_i} = 0, \qquad j = 0, 1,..., m - 1, \quad i = 1, 2,..., k. \tag{7.19}$$

Conversely, if $s \in U$ *satisfies all of the conditions* (7.16)–(7.19), *then it is a pLg-spline interpolating U.*

Proof. The necessity of (7.16), (7.17), and (7.18) follow from Theorem 5.1, Corollary 5.3, and Theorem 5.5, respectively. The smoothness assertion (7.19) is a consequence of Theorem 6.2 with the choice $l(\underline{x}_i) = l(\bar{x}_i) = 0$, $i = 1, 2, ..., k$.

We prove the converse by checking that for any $u \in U$, (3.1) holds. Let $g = u - s$. Then, using (7.16) and (7.18), just as in the proof of Theorems 7.1 and 7.2, we obtain

$$\int_a^b \theta Lg = \sum_{i=1}^{k} \int_{\underline{x}_i}^{\bar{x}_i} gL^*\theta$$
$$- \sum_{i=1}^{k} \sum_{j=0}^{m-1} (g^{(j)}(\bar{x}_i) \operatorname{jump}[L_j{}^*\theta]_{\bar{x}_i} + g^{(j)}(\underline{x}_i) \operatorname{jump}[L_j{}^*\theta]_{\underline{x}_i}).$$

In view of (7.19), all of the terms in the double sum vanish. Moreover,

$$\int_{\underline{x}_i}^{\bar{x}_i} gL^*\theta = K_i \int_{\underline{x}_i}^{\bar{x}_i} w_i(x)\, g(x)\, dx = K_i \lambda_i g.$$

But by (7.17), K_i and $\lambda_i g$ have the same sign, and (3.1) follows. ∎

8. Extension Problems

The problem of extending a function defined on a subset B of an interval I to the entire interval is of considerable imprtance in analysis. Of special interest are those extensions which are smooth, for example in some $L_p{}^m[I]$ space. Among the smooth extensions, one may ask for an optimal one in some sense. Golomb [14] has extensively studied the problem of optimal extensions in the sense that their mth derivative in the L_p-norm should be minimal. In this section we extend his work to optimal extensions with D^m replaced by an mth-order differential operator. The optimal extensions will be certain pLg-splines.

We begin with a precise definition of our extension problem. Let $B \subset I$, and suppose for each $x \in B$ that $l(x)$ is a positive integer with $l(x) \leqslant m$. Associate with each point $x \in B$ a set of real numbers $y_{(x,0)}, ..., y_{(x,l(x)-1)}$, and define the set

$$U = \{f \in L_p{}^m[I] : f^{(j)}(x) = y_{(x,j)}, j = 0, 1, ..., l(x) - 1, \text{ all } x \in B\}. \quad (8.1)$$

If F is some function defined on B with $F^{(j)}(x) = y_{(x,j)}, j = 0, 1, ..., l(x) - 1$, all $x \in B$, then any function $f \in U$ can be considered as an extension of F

to all of I. To define an optimal extension, let L be a linear mth-order differential operator as in (1.2). Then we say $s \in U$ is an *optimal extension* of F with respect to L provided it is a solution of (1.5), i.e., if s is a pLg-spline interpolating U.

The question of when a function F has an extension in $L_p{}^m[I]$ is equivalent to the question of when the U in (8.1) is nonempty. We do not attempt to answer this important question here. On the other hand, given that some extension exists, then by Theorem 2.1 (hypothesis (2.2) will be satisfied), we do know that there always exists an optimal extension. Thus, it is of interest to characterize pLg-splines interpolating U as in (8.1).

Before stating the main characterization result of this section, we need some further notation. We denote the derived set (set of cluster points) of B by B'. Then we define the *essential closure* of B to be the set

$$B_e = B' \cup \{x \in B\colon l(x) = m\}. \tag{8.2}$$

It is not hard to verify directly that B_e is a closed set.

To explain the terminology and usefulness of B_e, we note that $(a, b)\backslash B_e$ being an open set of real numbers, it can be written as a countable union of open intervals, say $(a, b)\backslash B_e = \bigcup_{i=1}^{\infty} J_i$ with $J_i = (\xi_i, \eta_i)$. The endpoints of each of these intervals must either belong to B_e or be one of the points a or b. It follows that with the possible exception of a and b, each of the points ξ_i and η_i is a point where the values of $f,..., f^{(m-1)}$ are all forced. (Indeed, either l is already m there, or the point belongs to B' and constraints on $f,..., f^{(m-1)}$ are implied—see Lemma 6.4.) We call the set $\Delta_e = \{\xi_i\}_1^{\infty} \cup \{\eta_i\}_1^{\infty}$ the set of *essential knots* associated with U. If x is an essential knot and $0 \leqslant j \leqslant m - 1$, then all the functions $f \in U$ have a common value of $f^{(j)}(x)$. We denote this value by $y_{(x,j)}$.

The following theorem is a complete characterization of optimal extensions.

THEOREM 8.1. *Given B and U defined as in* (8.1), *let* $\underline{x} = \inf B_e$ *and* $\bar{x} = \sup B_e$. *Then any pLg-spline interpolating U must satisfy the following conditions*:

there exists a function θ with $\theta = |\, Ls\,|^{p-1} \operatorname{sgn}(Ls)$ *a.e. such that*

$$L^*\theta = \quad \text{for all } x \in (a, b)\backslash B_e\,; \tag{8.3}$$

$$Ls = 0 \qquad \text{a.e. on } (a, \underline{x}) \text{ and } (\bar{x}, b); \tag{8.4}$$

$$\operatorname{jump}[L_j{}^*\theta]_x = 0, \qquad j = l(x),..., m - 1 \text{ and all } x \in B\backslash B_e\,; \tag{8.5}$$

$$s^{(j)}(x) = y_{(x,j)}\,, \qquad j = 0, 1,..., m - 1, \text{ all } x \in \Delta_e\,, \text{ where } \Delta_e \text{ is the set of of essential knots defined above.} \tag{8.6}$$

Conversely, suppose L has constant coefficients. Then, if $s \in U$ satisfies (8.3)–(8.6), *it is a pLg-spline interpolating U.*

Proof. Condition (8.6) is just the statement that s is completely specified at the essential knots, up to its $(m-1)$st derivative. The set of essential knots partitions the interval I into sets on which s is completely determined by the interpolation conditions, and subintervals $J_i = (\xi_i, \eta_i)$ where s must be the solution of the minimization problem:

$$\underset{u \in U_i}{\text{minimize}} \| Lu \|_{L_p[J_i]}, \tag{8.7}$$

where

$$U_i = \{f \in L_p{}^m[J_i]: f^{(j)}(x) = y_{(x,j)}, j = 0, 1,..., l(x) - 1, \\ \text{and all } x \in (B \cap J_i) \cup \{\xi_i\} \cup \{\eta_i\}\}. \tag{8.8}$$

(Here $l(\xi_i) = l(\eta_i) = m$, all i.) This, the characterization problem is reduced to solving the spline interpolation problems on each J_i. If J_i contains only a finite number of distinct points in B, then Theorem 6.1 serves to characterize s on J_i. It remains to consider the case where J_i contains an infinite set of distinct points from B (which then must have a cluster point at either ξ_i or η_i). This is handled in the following theorem. ▮

The following theorem charactizes pLg-splines which interpolate Hermite–Birkhoff data on a sequence of points converging to b.

THEOREM 8.2. *Let* $a < x_1 < x_2 < \cdots < x_\nu < \cdots < b$ *be a sequence converging (monotonely) to* b. *Let* $0 = \nu_{i,0} < \cdots < \nu_{i,l(x\cdot)-1} \leqslant m - 1$ *for* $i = 1, 2,\ldots$. *Let*

$$U = \{f \in L_p{}^m[I]: f^{(\nu_{ij})}(x_i) = y_{(x_i,j)}, \\ j = 0, 1,..., l(x_i) - 1 \textit{ and all } i = 1, 2,...\}, \tag{8.9}$$

where $y_{(x_i,j)}$ *are prescribed real numbers. Then any pLg-spline* s *interpolating* U *must satisfy the following conditions*:

$$\textit{For some } \theta \textit{ with } \theta = | Ls |^{p-1} \operatorname{sgn}(Ls) \quad \textit{a.e.}, L^*\theta = 0 \\ \textit{for all } x \in [a, b] \backslash \{x_i\}_1^\infty; \tag{8.10}$$

$$Ls = 0 \qquad \textit{a.e. on } (a, x_1); \tag{8.11}$$

$$\text{jump}[L_j{}^*\theta]_{x_i} = 0, \qquad j \in \{0, 1,..., m - 1\} \backslash \{\nu_{i,0},..., \nu_{i,l(x_i)-1}\}, \\ \textit{all} \quad i = 1, 2,...; \tag{8.12}$$

$$s^{(j)}(b) = y_{(b,j)}, \qquad j = 0, 1,..., m - 1, \textit{ where} \\ y_{(b,j)} \textit{ is the common value of all } f^{(j)}(b), f \in U. \tag{8.13}$$

Conversely, is L *has constant coefficients and* $s \in U$ *satisfies* (8.10)–(8.13), *then it is a pLg-spline interpolating* U.

Proof. The necessity of (8.10)–(8.12) follows from the results in Sections 5 and 6 in the usual way. Condition (8.13) follows from the implied constraints forced at b (cf. Lemma 6.4).

To prove the converse, we show (3.1) holds for any $u \in U$. Let $g = u - s$; then since s and g are in $L_p{}^m[I]$, it follows that $Lg \in L_p[I]$. Since $\theta \in L_q[I]$, we conclude that θLg is integrable, and so

$$\int_a^b \theta Lg = \lim_{\nu \to \infty} \int_a^{x_{N(\nu)}} \theta Lg$$

for any sequence $N(\nu)$ converging to ∞. Integrating by parts and using the properties (8.10)–(8.12), we obtain

$$\int_a^b \theta Lg = -\lim_{\nu \to \infty} \sum_{j=0}^{m-1} L_j{}^*\theta(x_{N(\nu)}) - g^{(j)}(x_{N(\nu)}).$$

Now, with $N(\nu)$ and δ_ν as in Lemma 6.4, we recall that

$$g^{(j)}(x_{N(\nu)}) = o(\delta_\nu^{m-j-1+1/q}), \qquad j = 0, 1,..., m-1.$$

It is proved below in Lemma 8.3 that

$$L_j{}^*\theta(x_{N(\nu)}) = o(\delta_\nu^{-(m-j-1+1/q)}), \qquad j = 0, 1,..., m-1.$$

Combining these facts, we conclude that $\int_a^b \theta Lg = o(1) \to 0$ as $\nu \to \infty$. We have established (3.1), and therefore that s is a pLg-spline interpolating U. ∎

The following technical lemma is required in the proof of Theorem 8.2 above. It is a direct analog of Lemma 5.1 of Golomb [14].

LEMMA 8.3. *Let $a \leqslant x_1 < x_2 < \cdots < x_\nu < \cdots < b$ be a sequence converging (monotonely) to b. Suppose that $\theta_\nu \in N_{L^*}$ on $(x_\nu, x_{\nu+1})$ for all ν (where L^* has constant coefficients), and that the function θ defined on $[a, b]$ by $\theta = \theta_\nu$ on $(x_\nu, x_{\nu+1})$ belongs to $L_q[a, b]$. Then for $j = 0, 1,..., m-1$,*

$$\| D^j\theta \|_{L_\infty[x_\nu, x_{\nu+1}]} = o(| x_{\nu+1} - x_\nu |^{-j-1/q}). \tag{8.14}$$

Thus, in particular,

$$\| L_j{}^*\theta \|_{L_\infty[x_\nu, x_{\nu+1}]} = o(| x_{\nu+1} - x_\nu |^{-(m-j-1+1/q)}), \tag{8.15}$$

for $j = 0, 1,..., m-1$.

Proof. Let $\{u_i\}_1^m$ be a basis for N_{L^*} on [0, 1]. Since L (and thus also L^*) has constant coefficients, a basis for N_{L^*} on any other interval is given by translation. Thus, we may write

$$\theta_\nu(x) = \sum_{i=1}^m a_{\nu i} u_i((x - x_\nu)/\epsilon_\nu) = \sum_{i=1}^m a_{\nu i} u_i(y)$$

with $y = (x - x_\nu)/\epsilon_\nu$, $\epsilon_\nu = (x_{\nu+1} - x_\nu)$. If $M = (\sum_{i=1}^m \| u_i^{(j)} \|_{L_\infty[0,1]}^p)^{1/p}$, we obtain

$$| D_x{}^j \theta_\nu(x)| \leqslant \left| \sum_{i=1}^m a_{\nu i} \left(\frac{1}{\epsilon_\nu}\right)^j u_i^{(j)}(y)\right| \leqslant M\epsilon_\nu^{-j} \| a_\nu \|_{l_q},$$

where a_ν is the vector $(a_{\nu 1},..., a_{\nu m})$ and $\| \cdot \|_q$ is the usual l_q-norm. The inequalities (8.14) follow if we show that $\| a_\nu \|_q = o(\epsilon_\nu^{-1/q})$. To do this, consider the linear mapping T: $N_{L^*}[0, 1] \to l_q(\mathbb{R}^m)$ defined by $T(\sum_1^m \alpha_i u_i) = (\alpha_1 ,..., \alpha_m)$. Since N_{L^*} and $\mathbb{R}^m$ are m-dimensional linear spaces, T is bounded; i.e., for some K,

$$\| a_\nu \|_q \leqslant K\left(\int_0^1 \left| \sum_1^m \int_0^1 a_{\nu i} u_i(y)^q \, dy\right)^{1/q} = K(\epsilon_\nu)^{-1/q} \| \theta_\nu \|_{L_q[x_\nu, x_{\nu+1}]}.$$

But $\| \theta_\nu \|_{L_q[x_\nu, x_{\nu+1}]} = o(1)$, and (8.14) is established.

The inequalities (8.15) follow from (8.14) if we take account of the fact that $L_j{}^* = \sum_{i=0}^{m-j-1} (-1) D^i a_{i+j+1}$ (cf. (6.2)). ∎

9. Examples

It is instructive to consider some examples illustrating the characterization theorems given above. We shall concentrate on Theorems 7.3 and 8.1 since examples illustrating the others can be found in the literature. (For an example with a finite number of EHB-linear functionals as in Theorem 7.1, see [16]. Examples where the spline is forced to lie between two prescribed functions as in Theorem 7.2 can be found in [5, 21], at least for $p = 2$.)

EXAMPLE 9.1. Find min $\int_0^5 [f'(x)]^2 \, dx$ over $U = \{f \in L_2{}^1[0, 5]: \int_1^2 f(x) \, dx = 1$ and $\int_3^4 f(x) \, dx = 2\}$.

Analysis and solution. Theorem 7.3 is applicable. First, (7.16) implies that a solution s must be linear on (2, 3), while (7.17) implies $L^*Ls = -s''$ must be constant (so s must be quadratic) on (1, 2) and (3, 4). Condition (7.18) implies s must be constant on the end intervals (0, 1) and (4, 5). To be in U, s must belong to $C[0, 5]$, but moreover, by (7.19) we must also have $L_0{}^*s = s'$

continuous at the knots 1, 2, 3, 4, so that in fact $s \in C^1[0, 5]$. This information suffices to construct s:

$$\begin{aligned} s(x) &= \tfrac{9}{10}, & 0 \leqslant x \leqslant 1,\\ &= \tfrac{9}{10} + 3(x-1)^2/10, & 1 \leqslant x \leqslant 2,\\ &= 3x/5, & 2 \leqslant x \leqslant 3,\\ &= \tfrac{21}{10} - 3(x-4)^2/10, & 3 \leqslant x \leqslant 4,\\ &= \tfrac{21}{10}, & 4 \leqslant x \leqslant 5. \end{aligned} \tag{9.1}$$

Since this function satisfies (7.16)–(7.19) and lies in U, it is a solution. Using Theorem 2.2, we easily check it is unique. ∎

EXAMPLE 9.2. Find $\min \int_0^3 [f'(x)]^2\,dx$ over $U = \{f \in L_2^1[0, 3]: 0 \leqslant \int_0^1 f(x)\,dx \leqslant 2,\ 3 \leqslant \int_1^2 f(x)\,dx \leqslant 4, \text{ and } 0 \leqslant \int_2^3 f(x)\,dx \leqslant 1\}$.

Analysis and solution. Theorem 7.3 applies. We deduce that a solution s must be piecewise quadratic, and $s \in C^1[0, 3]$. We also conclude from (7.19) that $\text{jump}[L_0{}^*s'] = \text{jump}[s'] = 0$ at the end points 0 and 3. This information permits the construction of (the unique solution):

$$\begin{aligned} s(x) &= 6x^2/5 + \tfrac{8}{5}, & 0 \leqslant x \leqslant 1,\\ &= -3x^2 + 42x/5 - \tfrac{13}{5}, & 1 \leqslant x \leqslant 2,\\ &= 9x^2/5 - 54x/5 + \tfrac{83}{5}, & 2 \leqslant x \leqslant 3. \end{aligned} \tag{9.2}$$

The solution of Example 9.2 is a kind of histospline (cf. [2, 3, 25]). The minimization problem could have been solved by converting it to one involving point evaluation functionals (cf. [3]). This is not the case, however, with the problem in Example 9.1.

We conclude with an example involving optimal extensions of functions.

EXAMPLE 9.3. Find $\min \int_0^4 [f''(x)]^2\,dx$ over $U = \{f \in L_2^2[0, 4]: f(t) = t^2(t-2)(t-3)$ for $t \in (1, 2) \cup (3, 4) \cup \{\tfrac{1}{2}\} \cup \{\tfrac{5}{2}\}$ and $f'(\tfrac{1}{2}) = \tfrac{3}{2}\}$.

Analysis and solution. We use Theorem 8.1. The set $B_e = \{\tfrac{1}{2}\} \cup [1, 2] \cup [3, 4]$ in this case, and the set of essential knots is $\Delta_e = \{\tfrac{1}{2}, 1, 2, 3, 4\}$ in this case. Here $m = 2$, $L = D^2 = L^*$, and $L_0{}^* = I$, $L_1{}^* = -D$. We conclude that s must be linear in $(0, \tfrac{1}{2})$ and cubic on the intervals $(\tfrac{1}{2}, 1)$, $(2, \tfrac{5}{2})$, $(\tfrac{5}{2}, 3)$. Globally, $s \in C^1[0, 4]$, and the cubic pieces must join at $\tfrac{5}{2}$ with two continuous derivatives. This information permits the construction of the unique solution:

$$\begin{aligned} s(x) &= 3x/2 + \tfrac{3}{16}, & 0 \leqslant x \leqslant \tfrac{1}{2}\\ &= \tfrac{15}{16} + 2(x-\tfrac{1}{2})/2 + 19(x-\tfrac{1}{2})^2/4 - 7(x-\tfrac{1}{2})^3, & \tfrac{1}{2} \leqslant x \leqslant 1\\ &= x^2(x-2)(x-3), & 1 \leqslant x \leqslant 2\\ &= -4(x-2) - (x-2)^2/4 + 4(x-2)^3, & 2 \leqslant x \leqslant \tfrac{5}{2}\\ &= 9(x-3) + 59(x-3)^2/4 + 6(x-3)^3, & \tfrac{5}{2} \leqslant x \leqslant 3\\ &= x^2(x-2)(x-3), & 3 \leqslant x \leqslant 4. \end{aligned} \tag{9.3}$$

10. Notes and Remarks

1. We have considered only the case $1 < p < \infty$ since for p in this range, the space $L_p[I]$ is uniformly convex. The cases of $p = 1$ and $p = \infty$ have recently attracted considerable attention; e.g., see [4, 7–10].

2. The interval I was restricted to be a finite closed interval throughout this paper. It is possible to consider pLg-splines defined on unbounded intervals provided the space L_p^m is properly defined (cf. Golomb [14] and Smith [26]).

3. The tools developed here can also be used to characterize the structure of certain smoothing pLg-splines. To define these, let $\tilde{A}$ be a finite collection of linear functionals as in Section 1, and let $\{y_\alpha\}_{\tilde{A}}$ be a set of given data. With positive weights $\{w_\alpha\}_{\tilde{A}}$ define

$$\sigma_{w,y}(u) = \| Lu \|_p^p + \sum_{\alpha \in \tilde{A}} w_\alpha \mid \lambda_\alpha u - y_\alpha \mid^p.$$

We call a function $s \in U$ a smoothing pLg-spline provided it satisfies

$$\sigma_{w,y}(s) = \inf_{u \in U} \sigma_{w,y}(u).$$

The structure of smoothing pLg-splines can then be read off from the fact that they are completely characterized by the condition that

$$0 \leqslant \int_a^b \mid Ls \mid^{p-1} \operatorname{sgn}(Ls)(Lu - Ls)$$

$$+ \sum_{\alpha \in \tilde{A}} w_\alpha \mid \lambda_\alpha s - y_\alpha \mid^{p-1} \operatorname{sgn}(\lambda_\alpha s - y_\alpha) \cdot (\lambda_\alpha u - y_\alpha)$$

for all $u \in U$. For some results with $p = 2$, see Nielson [22].

4. Lemmas 4.1 and 4.2 can be found in the literature on distributions. The first part of Lemma 4.1 is a result on Radon measures (cf. Treves [29, Theorem 21.3]). The last part is a theorem of Bremmerman [27, p. 16]. Lemma 4.2 can be derived from the fact that every distribution on I is of finite order (cf. Halperin [28]). We are grateful to the referee for these references. The proofs of these results are sufficiently short, however, that it may be of some value to include them here, which we do in the following two remarks.

5. *Proof of Lemma* 4.1. For the first part, suppose $f(x) > 0$ on a set $J \subset (c, d)$ of positive measure. Then $|f|^{p-1} \chi_J \in L_q[c, d]$, where $1/p + 1/q = 1$. Since $C^\infty(c, d)$ is dense in $L_q[c, d]$, we can find a sequence

$\varphi_\nu \in C^\infty(c, d)$ of nonnegative functions converging to $|f|^{p-1} \chi_J$ in L_q. This yields the contradiction

$$0 < \int_J f^p(x)\, dx = \int_c^d f(x)\, |f(x)|^{p-1} \chi_J(x)\, dx = \lim \int_c^d f(x)\, \varphi_\nu(x)\, dx = 0.$$

To prove the second statement, let D_c^{-1} be defined by $D_c^{-1} f(x) = \int_c^x f(t)\, dt$, and let $D_c^{-j} = D_c^{-1} D_c^{-j+1}$. Then, given $\psi \in C^\infty(c, d)$, let $\varphi = D_c^{-m}\psi$. Clearly, $\varphi \in C_{m-1}^\infty(c, d)$, and by the hypothesis (4.7), since $D^m\varphi = \psi$, it follows $\int_c^d f\psi = 0$ for all $\psi \in C^\infty(c, d)$. By the first part of the lemma, $f = 0$ a.e. on (c, d). The proof for $\bar{C}_{m-1}^\infty(c, d)$ is similar.

Finally, we prove part 3 of the lemma. Assume (4.8) holds. We shall use the shorthand $(f, g) = \int_c^d fg$. Let $p_f \in \mathscr{P}_m$ be chosen so that $(f - p_f, x^i) = 0$, $i = 0, 1, \ldots, m - 1$. In view of Lemma 4, it suffices to show that $(f - p_f, \psi) = 0$ for all $\psi \in C^\infty(c, d)$. Given $\psi \in C^\infty(c, d)$, choose $q_\psi \in \mathscr{P}_m$ so that $(\psi - q_\psi, x^i) = 0$, $i = 0, 1, \ldots, m - 1$. Define

$$\varphi(x) = \int_c^d \frac{(x - t)_+^{m-1}}{(m - 1)!} [\psi - q_\psi](t)\, dt.$$

Then clearly $\varphi^{(j)}(c) = 0$, $j = 0, 1, \ldots, m - 1$. Moreover, since $\psi - q_\psi$ is orthogonal to $\mathscr{P}_m$, we also have $\varphi^{(j)}(d) = 0$, $j = 0, 1, \ldots, m - 1$. Thus, $\varphi \in C_{m-1}^\infty(c, d)$. Using the orthogonality of both $f - p_f$ and $\psi - q_\psi$ to $\mathscr{P}_m$, we have

$$(f - p_f, \psi) = (f - p_f, \psi - q_\psi) = (f, \psi - q_\psi) = (f, \varphi^{(m)}) = 0.$$

6. *Proof of Lemma* 4.2. Let D_c^{-j} be the operators in the proof of Lemma 4.1. Integrating (4.11) by parts, we obtain

$$\int_c^d \varphi(\mu) \sum_{j=0}^{\mu} (-1)^{\mu-j} D_c^{j-\mu}(b_j f) = 0, \qquad \text{all } \varphi \in C_{\mu-1}^\infty(c, d).$$

Lemma 4.1 asserts that the sum is equal to a polynomial of degree $\mu - 1$ a.e. on (c, d). Thus if we modify f on a set of measure 0, we obtain a function θ_f such that

$$\sum_{j=0}^{\mu} (-1)^{\mu-j} D_c^{j-\mu}(b_j \theta_f) = p_f \in \mathscr{P}_\mu$$

for all $x \in I$. Differentiating μ times, we obtain

$$\sum_{j=0}^{\mu} (-1)^j D^j(b_j \theta_f) = 0.$$

We conclude $\theta_f \in N_{M^*}$, and the lemma is proved.

REFERENCES

1. P. M. ANSELONE AND P. J. LAURENT, A general method for the construction of interpolating or smoothing spline functions, *Numer. Math.* **12** (1968), 66–82.
2. L. BONEVA, D. KENDALL, AND I. STEFANOV, Spline transformations, *J. Royal Statist. Soc. Ser. B* **33** (1971), 1–70.
3. C. DE BOOR, Appendix to Splines and histograms" by I. J. Schoenberg, *in* "Spline Functions and Approximation Theory" (A. Meir and A. Sharma, Eds.), ISNM Vol. 21, pp. 329–358, Birkhauser, Basel, 1973.
4. C. DEBOOR, A remark concerning perfect splines, *Bull. Amer. Math. Soc.* **80** (1974), 723–727.
5. P. A. COPLEY, "Structure and Characterization of pLg-Splines," Dissertation, University of Texas at Austin, 1974.
6. J. W. DANIEL AND L. L. SCHUMAKER, On the closedness of the linear image of a set, with applications to generalized spline functions, *Appl. Anal.* **4** (1974), 191–205.
7. S. D. FISHER AND J. W. JEROME, The existence, characterization, and essential uniqueness of L_∞ minimization problems, *Trans. Amer. Math. Soc.* **187** (1974), 391–404.
8. S. D. FISHER AND J. W. JEROME, Perfect spline solutions to L_∞ extremal problems, *J. Approximation Theory* **12** (1974), 78–90.
9. S. D. FISHER AND J. W. JEROME, Spline solutions to L_1 extremal problems in one and several variables, *J. Approximation Theory* **13** (1975), 73–83.
10. S. D. FISHER AND J. W. JEROME, "Minimum Norm Extremals in Function Spaces," Lecture Notes in Mathematics 479, Springer–Verlag, Berlin, 1975.
11. I. N. GELFOND AND G. E. SHILOV, "Generalized Functions," Academic Press, New York, 1964.
12. J. R. GILES, Classes of semi-inner-product spaces, *Trans. Amer. Math. Soc.* **129** (1967), 436–446.
13. M. GOLOMB, "Splines, n-Widths, and Optimal Approximation," MRC Rep. 784, Univ. of Wisconsin, 1967.
14. M. GOLOMB, $H^{m,p}$-extensions by $H^{m,p}$-splines, *J. Approximation Theory* **7** (1972), 238–275.
15. M. GOLOMB AND J. W. JEROME, Linear ordinary differential equations with boundary conditions on arbitrary point sets, *Trans. Amer. Math. Soc.* **153** (1971), 235–264.
16. J. W. JEROME, Linearization in certain nonconvex minimization problems and generalized spline projections, *in* "Spline Functions and Approximation Theory," ISNM Vol. 21, pp. 119–166, Birkhäuser-Verlag, Basel.
17. J. W. JEROME AND L. L. SCHUMAKER, On Lg-splines, *J. Approximation Theory* **2** (1969), 29–49.
18. J. W. JEROME AND L. L. SCHUMAKER, Characterization of functions with higher-order derivatives in L_p, *Trans. Amer. Math. Soc.* **143** (1969), 363–371.
19. J. W. JEROME AND L. L. SCHUMAKER, Characterizations of absolute continuity and essential boundedness for higher-order derivatives, *J. Math. Anal. Appl.* **42** (1973), 452–465.
20. J. W. JEROME AND R. S. VARGA, Generalizations of spline functions and applications to nonlinear boundary-value problems, and eigenvalue problems, *in* "Theory and Applications of Spline Functions" (T. N. E. Greville, Ed.), pp. 103–155, Academic Press, New York, 1969.
21. O. L. MANGASARIAN AND L. L. SCHUMAKER, Splines via optimal control, *in* Approximation with Special Emphasis on Spline Functions (I. J. Schoenberg, Ed.), pp. 119–156, Academic Press, New York, 1969.
22. G. M. NIELSON, Multivariate smoothing and interpolating splines, *SIAM J. Numer. Anal.* **11** (1974), 435–446.

23. H. ROYDEN, "Real Analysis," Macmillan, New York, 1963.
24. A. SARD, Integral representations of remainders, *Duke Math. J.* **15** (1948), 333–345.
25. I. J. SCHOENBERG, Splines and Histograms, *in* "Spline Functions and Approximation Theory" (A. Meir and A. Sharma, Eds.), ISNM Vol. 21, pp. 277–327, Birkhäuser, Basel, 1973.
26. P. W. SMITH, $W^{r,p}$ splines, *J. Approximation Theory* **10** (1974), 337–357.
27. H. BREMERMANN, "Distributions, Complex Variables, and Fourier Transforms," Addison-Wesley, Reading, Mass., 1965.
28. I. HALPERIN, "Introduction to the Theory of Distributions," Univ. of Toronto Press, Toronto, 1952.
29. F. TREVES, "Topological Vector Spaces, Distributions, and Kernels," Academic Press, New York, 1967.